

The 4-dimensional Taub string

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Abstract

The prototype of a Taub string is formed by successive junctions of copies of Taub's space \mathcal{T} , joined at their null boundaries Σ to create the axially-symmetric Bianchi-type-XI (with compact SL sections of scale L_o) vacuum $\mathcal{B}_T^4 = \cdots \vee \mathcal{T} \vee \mathcal{T} \vee \mathcal{T} \vee \cdots$, which is a *proper* one, namely a stable, non-singular, geodesically and globally fit solution of Einstein's vacuum equations without torsion and without a cosmological constant. Each \mathcal{T} contributes to \mathcal{B}_T^4 with its entire life-span as a quantum of time $\delta t \sim L_o$ between two consecutive Σ . The latter propagate as shock-wave fronts under string tension of Planck-scale strength κ_o . The incurring dynamics entails stability and the foundation of hierarchy in \mathcal{B}_T^4 . Appropriate averaging of this dynamics generates *effective* stress-energy content and torsion in a static $\bar{\mathcal{B}}_T^4$ vacuum. With the latter as ground state, excitations thereof must involve two new independent scales, κ and $L_1 \gg L_o$, in addition to κ_o , L_o . Elemental finite Taub strings and variant vacua, including the $L_o \gg \kappa_o$ cosmological case, are also discussed.

1 Introduction

The elegance in current geometrical settings, mitigated as it is by fundamental problems on stability and hierarchy, offers strong motivation for the search of new perspectives in context. At the accordingly fundamental level of a spacetime manifold, we will resort to the notions of a *proper vacuum*¹ \mathcal{V} and of an *effective proper vacuum*² $\bar{\mathcal{V}}$, aiming to search beyond Minkowski's \mathcal{M}_0^4 (which, modulo pp-waves, is the only 4D proper vacuum with open SL sections) to find all possible 4D proper vacua with *compact* SL sections.

To do that, one must exhaust the classification of homogeneous spacetimes [1], [2], [3], to come-up with a generic Bianchi-type IX, the $\mathcal{B}_{\text{IX}}^4$, as the only such candidate. Its 3D orthochronous sections, of any scale L_0 , exhibit the same dynamics of $SU(2)$ acting transitively on its own group manifold, the homogeneous S^3 . The minimally symmetric case, categorized as (3+0) for its three transitive Killing vectors, is, of course, Misner's mixmaster³ \mathcal{B}_{M}^4 [4]. Probably non-singular, \mathcal{B}_{M}^4 is certainly beyond analytic treatment except for exact or effective symmetries, such as those due to chaotic mixing over scales sufficiently larger than L_0 . The maximally-symmetric (3+3) case is incompatible with the vacuum equations and the (3+2) case cannot exist either (the two generators cannot form a subgroup of isometries). The remaining (3+1) case is Taub's axially-symmetric solution \mathcal{T} [5], often treated as if it were singular, which it is not; \mathcal{T} does possess an initial and a final *physical* singularity, namely its pair of null boundaries Σ , but at such singularities (in contrast to mathematical ones) the volume element and the entire Riemann tensor must remain finite. Regardless of coordinate failure on the Σ , the problem is with extending the geodesics beyond those boundaries: it is a pathogeny to be cured, otherwise \mathcal{T} will remain only part of a missing whole. The Taub-NUT construction does cure it [6] but still fades as a proper vacuum on certain global aspects, and so does any finite number of \mathcal{T} joined in a closed or open string formation⁴. We have thus been led to the axially symmetric proper 4D vacuum

$$\mathcal{B}_{\text{T}}^4 = \bigvee_{n=-\infty}^{n=+\infty} \mathcal{T}_n = \cdots \vee \mathcal{T}_{n-1} \vee \mathcal{T}_n \vee \mathcal{T}_{n+1} \cdots, \quad n \in \mathbb{Z}, \quad (1.1)$$

¹Namely a solution of Einstein's vacuum equations without torsion and without a cosmological constant which is mathematically and physically acceptable, namely non-singular, stable, geodesically and globally fit, especially with no closed TL curves due to vorticity, chaoticity or topology beyond Planck scale.

²A non-static \mathcal{V} with compact SL sections could, under time averaging, descend to a static $\bar{\mathcal{V}}$ with acquired features, such as effective energy-momentum and torsion content, as relics of the original dynamics.

³The term relates to the non-linear mixmaster dynamics, originally proposed as a homogenization and isotropization mechanism for the study of horizons and cosmic mixing in relativistic cosmology.

⁴The Taub-NUT space is formed by a \mathcal{T} sandwiched via two Σ junctions (a pair of 'Misner bridges') between two NUT spaces as $\text{NUT} \vee \mathcal{T} \vee \text{NUT}$. In the present construction we use only Taub copies in consecutive junctions, indefinitely, as $\cdots \vee \mathcal{T} \vee \mathcal{T} \vee \mathcal{T} \vee \cdots$, for a likewise non-singular result.

as prototype of a Taub string, essentially unique, given the elusive nature of Misner's \mathcal{B}_M^4 . The integer n enumerates the \mathcal{T}_n and their boundaries Σ_n (null squashed S^3) at the junctions. The periodic metric of \mathcal{B}_T^4 is expressible in terms of the $b = b(u)$, $c = c(u)$ radii (cf. also Fig.1 in section 2) as functions of u , which is a null coordinate, namely with $(\partial_u)^2 = 0$, and of the θ, ϕ, ψ coordinates on the homogeneous $du = 0$ squashed- S^3 hypersurfaces as

$$ds^2 = -2L_o du(d\psi + \cos\theta d\phi) + L_o^2 (b^2[(d\theta)^2 + \sin^2\theta(d\phi)^2] + c^2(d\psi + \cos\theta d\phi)^2) . \quad (1.2)$$

The θ, ϕ, ψ also define the $SU(2)$ left-invariant ℓ^μ ($\ell^0 = du$, $\ell^1/L_o = \cos\psi d\theta + \sin\theta \sin\psi d\phi$, $\ell^2/L_o = -\sin\psi d\theta + \sin\theta \cos\psi d\phi$, $\ell^3/L_o = \cos\theta d\phi + d\psi$ for $d\ell^i = -\frac{1}{2L_o}\epsilon_{jk}^i \ell^j \wedge \ell^k$) with dual L_μ ($L_0 = \partial_u$, L_i), and $g_{03} = -1$, $g_{11} = g_{22} = b^2$, $g_{33} = c^2$ in a non-holonomic equivalent of (1.2) as

$$ds^2 = -2du(\ell^3) + b^2[(\ell^1)^2 + (\ell^2)^2] + c^2(\ell^3)^2. \quad (1.3)$$

As we will see, in-between the boundaries Σ_n, Σ_{n+1} of \mathcal{T}_n , the $du = 0$ sections in (1.3) remain SL while being transported by ∂_u but they become momentarily null when they arrive at (and identify with) Σ_{n+1} . The latter is the final boundary of \mathcal{T}_n and simultaneously the initial of \mathcal{T}_{n+1} . Each \mathcal{T} contributes to \mathcal{B}_T^4 with its entire life-span as a quantum of time $\delta t \sim L_o$ between two consecutive Σ , which propagate as shock-wave fronts under string tension of (roughly) Planck-scale κ_o strength. The incurring dynamics entails stability and the foundation of hierarchy in \mathcal{B}_T^4 . Appropriate averaging of this dynamics generates *effective* stress-energy content, possibly from effective torsion in a static $\bar{\mathcal{B}}_T^4$ vacuum. We'll also discuss variant vacua including finite Taub strings and the cosmological $L_o \gg \kappa_o$ case.

Notation & conventions: The metric in $ds^2 = g_{\mu\nu}\ell^\mu\ell^\nu$, with $dg_{\mu\nu} = \Gamma_{\mu\nu} + \Gamma_{\nu\mu}$, simplifies to $\eta_{\mu\nu} = (-1, +1, +1, +1)$ in orthonormal Cartan coframes $\theta^\mu = \theta^\mu_\nu \ell^\nu$ with dual Θ_μ ; the general connection γ^μ_ν and the Christoffel $\Gamma^\mu_\nu = \Gamma^\mu_{\nu\rho}\theta^\rho$ (with covariant derivatives \mathcal{D} , D , respectively) are antisymmetric in μ, ν just like the contorsion tensor-valued 1-form K^μ_ν in

$$\gamma^\mu_\nu = \Gamma^\mu_\nu + K^\mu_\nu, \quad D\theta^\mu := d\theta^\mu + \Gamma^\mu_\nu \wedge \theta^\nu \equiv 0, \quad \mathcal{D}\Theta_\mu = d\Theta_\mu - \Gamma^\nu_\mu \mathcal{L}_\nu \equiv 0 . \quad (1.4)$$

The general curvature \mathcal{R}^μ_ν includes its Riemannian part $R^\mu_\nu := d\Gamma^\mu_\nu + \Gamma^\mu_\rho \wedge \Gamma^\rho_\nu$, with $W^\mu_{\nu\rho\sigma}$ and $R_{\mu\nu} = R^\rho_{\mu\rho\nu}$ the Weyl and Ricci tensors. Cartan's first and second structure equations involve the general curvature \mathcal{R}^μ_ν and the torsion tensor-valued 2-form T^μ as [7]

$$\mathcal{R}^\mu_\nu : = d\gamma^\mu_\nu + \gamma^\mu_\rho \wedge \gamma^\rho_\nu = R^\mu_\nu + DK^\mu_\nu + K^\mu_\rho \wedge K^\rho_\nu = \frac{1}{2}\mathcal{R}^\mu_{\nu\rho\sigma}\theta^\rho \wedge \theta^\sigma, \quad (1.5)$$

$$T^\mu : = \mathcal{D}\theta^\mu = d\theta^\mu + \gamma^\mu_\nu \wedge \theta^\nu = K^\mu_\nu \theta^\nu = \frac{1}{2}T^\mu_{\rho\sigma}\theta^\rho \wedge \theta^\sigma . \quad (1.6)$$

The geodesic and Killing (by the \mathcal{L}_Ξ Lie derivative) equations are employed as

$$U = U^\mu \Theta_\mu : \quad \nabla_U U = 0 \longrightarrow DU^\mu(U) = 0 \text{ [extending to] } \mathcal{D}U^\mu(U) = 0, \quad (1.7)$$

$$\Xi = \Xi^\mu \Theta_\mu : \quad \mathcal{L}_\Xi \theta^\mu = d\Xi^\mu + (\Gamma^\mu_{\rho\nu} - \Gamma^\mu_{\nu\rho})\Xi^\rho \theta^\nu \longrightarrow D\Xi^\mu = \Gamma^\mu_{\nu\rho}\Xi^\rho \theta^\nu, \quad (1.8)$$

with $U \cdot U = U_\mu U^\mu = \varsigma = 0, \mp 1$ for the null, TL or SL cases.

2 The \mathcal{B}_T^4 proper vacuum as a Taub string

Taub's general solution for \mathcal{T} , to be expressed in terms of the $b = b(u)$, $c = c(u)$ functions in (1.3), needs a subtle refinement for compatibility at every junction across the Σ_n boundaries in (1.1) and thence for the entire \mathcal{B}_T^4 . In addition to (ℓ^μ, L_ν) , we will also employ two different sets of *orthonormal* Cartan frames in \mathcal{B}_T^4 , the (θ^μ, Θ_ν) and the (e^μ, E_ν) , with

$$\theta^0 = c^{-1}du, \quad \theta^1 = b\ell^1, \quad \theta^2 = b\ell^2, \quad \theta^3 = c\ell^3 - c^{-1}du; \quad (2.1)$$

$$\Theta_0 = c\partial_u + c^{-1}L_3, \quad \Theta_1 = b^{-1}L_1, \quad \Theta_2 = b^{-1}L_2, \quad \Theta_3 = c^{-1}L_3,$$

$$e^0 = \frac{1}{\sqrt{2}} \left[du + \left(1 - \frac{c^2}{2}\right)\ell^3 \right], \quad e^1 = b\ell^1, \quad e^2 = b\ell^2, \quad e^3 = \frac{1}{\sqrt{2}} \left[du - \left(1 + \frac{c^2}{2}\right)\ell^3 \right], \quad (2.2)$$

$$E_0 = \frac{1}{\sqrt{2}} \left[\left(1 + \frac{c^2}{2}\right)\partial_u + L_3 \right], \quad E_1 = b^{-1}L_1, \quad E_2 = b^{-1}L_2, \quad E_3 = \frac{1}{\sqrt{2}} \left[\left(1 - \frac{c^2}{2}\right)\partial_u - L_3 \right],$$

which are quite useful in spite of the failure of (θ^μ, Θ_ν) on Σ_n and the non-holonomic time in (e^μ, E_ν) . The Christoffel 1-forms $\Gamma^0_1 = c(\ln b)\theta^1$, $\Gamma^0_3 = \dot{c}\theta^3$, etc., with a dot for d/du , provide the Riemann and Weyl tensors (finite everywhere) in the (θ^μ, Θ_ν) frames as

$$R^0_{101} = R^0_{202} = c^2(\ln b)'' + c^2(\ln b)'^2 + c\dot{c}(\ln b)', \quad R^0_{303} = c\ddot{c} + \dot{c}^2 \quad (2.3)$$

$$R^2_{323} = R^3_{131} = c\dot{c}(\ln b)' + \frac{c^2}{4L_o^2 b^4}, \quad R^1_{212} = c^2(\ln b)'^2 + \frac{4b^2 - 3c^4}{4L_o^2 b^4} \quad (2.4)$$

$$R^0_{123} = R^0_{231} = -\frac{1}{2}R^0_{312} = W^0_{123} = W^0_{231} = -\frac{1}{2}W^0_{312} = \frac{1}{4L_o} \left(\frac{c^2}{b^2} \right)'. \quad (2.5)$$

In addition to the basic scale L_o from the frames, Taub's general solution of $R_{\mu\nu} = 0$ has two independent constant parameters, $B > 0$ and $N \in \mathbb{R}$, essentially from time scaling and translational invariance, respectively. However, that solution may also be expressed as

$$b^2 = \frac{1}{2B} \left(1 + 4 \left(\frac{u}{L'_o} \right)^2 \right), \quad c^2 = \frac{2}{B} \cdot \frac{1 - 4(u/L'_o)^2}{1 + 4(u/L'_o)^2}, \quad \left[L'_o = \frac{2L_o}{B} \right], \quad (2.6)$$

and still remain the general one, in spite of the absence of N . The latter is hidden in the allowed $u \rightarrow u + NL'_o$ translations (with $N \in \mathbb{R}$, $L'_o = 2L_o/B$), under which (2.6) is *not* form invariant but remains a solution of $R_{\mu\nu} = 0$. This will be utilized as a tool to reform the general Taub solution (*with* the $N \in \mathbb{R}$) in (2.6), to also being, simultaneously, the L'_o -periodic solution for \mathcal{B}_T^4 in (1.1). However, in the second interpretation, $N \in \mathbb{R}$ no longer exists as an independent parameter because, restricted to its integer values as $N \in \mathbb{Z}$, it has been consumed as a 'summation' index n in (1.1), as depicted in Fig.1. Thus, the original translational invariance, with (2.6) as the Taub general solution, has been traded

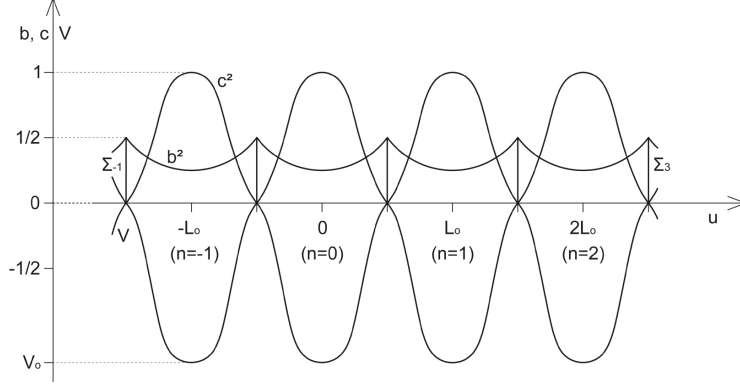


Figure 1: The Taub string \mathcal{B}_T^4 , depicted in terms of its radii in the metric (1.2), with the b^2 , c^2 of (2.6) in units of L_o (with $B = 2$), along the potential wells $V = V(u)$. The short vertical lines represent the propagating null squashed- S^3 junction hypersurfaces Σ_{-1} to Σ_3 .

for periodicity in (1.1), now viewed as the \mathcal{B}_T^4 solution. The ensuing quantization of the time coordinate is crucially involved in stability and hierarchy aspects, interrelated via string tension of a κ_o^2 strength, as we will see. We still have, of course, $(\partial_u)^2 = 0$, $(\partial_u) \cdot (L_3)^2 = -1$, $(L_3)^2 = c^2$ so, by the latter, we verify that L_3 turns momentarily null and orthogonal to the Σ_n at the $\pm L_o'/2$ roots of $c^2 = 0$, as follow from (2.6). The entire life-time span of \mathcal{T} , as it extends in-between these roots, constitutes an elemental ‘quantum of time’ contribution to \mathcal{B}_T^4 . Its proper-time duration δt is actually calculable along TL geodesics in terms of the u coordinate in the $-L_o'/2 \leq u \leq L_o'/2$ interval. The latter, now realized as the $n = 0$ one in (1.1), extends periodically as $[n - L_o'/2, n + L_o'/2]$ for any $n \in \mathbb{Z}$, as also depicted in Fig. 1. At the extremes of these intervals, where the physical singularities Σ_n of the \mathcal{T}_n are located, the volume element never vanishes and the components of R^μ_ν never diverge.

Our less-than- C^2 junctions at the Σ_n boundaries in \mathcal{B}_T^4 of (1.1) may (and they do) carry surface tensions, calculable from C^0 -junction compatibility conditions. The normal to the propagating $du = 0$ hypersurfaces is the TL unit vector $\mathcal{N} = \Theta_0$, but these hypersurfaces become null at $c = 0$, where they reach (and identify with) the Σ . The normal to the latter is the (momentarily null there) L_3 vector, so the extrinsic curvature vector-valued 1-form is $\mathcal{K} = -dL_3$. The discontinuity (jump) $\delta\mathcal{K}$ across every Σ reveals a stress-energy layer thereon, which creates string tension across every Σ along \mathcal{B}_T^4 . To see that, we can start the computation of the stress-energy tensor $S_{\mu\nu}$ layers with $\mathcal{N} = L_3 = c\Theta_3$ in the (θ^μ, Θ_ν) frame of (2.1) and, after cancellation of the anomalies at $c = 0$, switch back to the (ℓ^μ, L_ν) . We thus find that the discontinuities $\delta\mathcal{K} = -\delta d(c\Theta_3)$ occur only on the $(0,3)$ plane across Σ and only as $\delta\mathcal{K}_3^0 = \delta\mathcal{K}_0^3 = -\frac{4}{L_o}(c^2)$. The full result for the $S_{\mu\nu}$ layers, expressed in either of the

$(\ell^\mu, L_\nu), (e^\mu, E_\nu)$ frames (the θ^μ, Θ_ν fail on Σ_n), involves the only non-vanishing

$$S_{00} = S_{33} = +\frac{2B}{L_o^2 \kappa_o^2} \sum_{n=-\infty}^{n=\infty} \delta\left(u - (n + \frac{1}{2})L'_o\right), \quad n \in Z, \quad (2.7)$$

components of a $S_{\mu\nu} \sim \text{diag}(1, 0, 0, 1)$ layer of Planck-scale κ_o^2 strength, traceless and with correct content and sign for string tension orthogonal to the L_1, L_2 directions. This self-consistent (with no external sources) geometric disturbance (which cannot be a pp-wave) occurs across the compact null interface Σ and involves the latter as a gravitational shock-wave front [8]. Thus, the stability of the dynamics in-between the Σ_n is being extended across them and over the entire \mathcal{B}_T^4 manifold⁵ (cf. also discussion in the last section).

For the $U = U^\mu \Theta_\mu$ tangent to geodesics in \mathcal{B}_T^4 , with $u = u(t), \chi = \chi(t)$ in terms of the proper-time parameter t and $du/dt = 1/\dot{t}$ (P_\perp, E are constants of motion), we find

$$U^0 = \frac{1}{c\dot{t}}, \quad U^1 = P_\perp \frac{\cos \chi}{b}, \quad U^2 = -P_\perp \frac{\sin \chi}{b}, \quad U^3 = \frac{\sqrt{2E}}{c}, \quad (2.8)$$

plus two first integrals expressible in terms of the given $v = v(u)$ function as

$$\dot{\chi} = \frac{c^2 - b^2(1 + \cosh v)}{L_o b^2 c^2 \cosh v}, \quad \sinh^2 v := \frac{1}{2E} \left(\frac{P_\perp^2}{b^2} - \varsigma \right) c^2, \quad (2.9)$$

$$\dot{t} = \frac{1}{\sqrt{2E} \cosh v} \longleftrightarrow \frac{1}{2} \left(\frac{du}{dt} \right)^2 + V(u) = E, \quad V(u) = -E \sinh^2 v. \quad (2.10)$$

The so-emerging $n = 0$ potential-well function of $V = V(u)$, as shown in Fig.1, is a known stability aspect of the dynamics within \mathcal{T} . It has a minimum of $-(2P_\perp^2 + 1/B)$ at $u = 0$, with s-c (*supremum*-case) value of $V_o = -1/B$ if $P_\perp = 0$, and TL geodesics (with $E < 0$) trapped within the wells of all n . The latter aspect, not quite the same as geodesic inextendibility (especially if it is involved at Planck scale), may survive our \mathcal{B}_T^4 completion of \mathcal{T} , as in the Taub-NUT case. In the general dynamics from (2.10), all $E > 0$ geodesics in the singularity-free geometry of \mathcal{B}_T^4 propagate freely, except for an instantaneous impact at the junctions, due to stresses on the Σ boundaries. Nevertheless, we do have peculiar behavior in the case of s-c geodesics. They certainly form a marginal subclass (due to the vigorous $P_\perp = 0$ requirement), but the dynamics can be easily integrated in that case to the elegant result

$$U^0 = \sqrt{1 + \frac{2E}{c^2}} = \gamma, \quad U^3 = \sqrt{\frac{2E}{c^2}} = \beta\gamma \quad \left[\gamma = \frac{1}{\sqrt{1 - \beta^2}}, \quad \beta^2 = \frac{2E}{c^2 + 2E} \right], \quad (2.11)$$

also expressed here in terms of the special-relativistic β and γ parameters on the (Θ_0, Θ_3) plane. Expectedly, all s-c geodesics turn momentarily null as they cross the Σ boundaries, where $\beta \rightarrow 1$ as $c \rightarrow 0$, but with complete ‘loss of memory’ of whatever $E > 0$ value they had, due to its cancellation (an unexpected result we will return to).

⁵The synergy of breathing-mode dynamics with the $V(u)$ potential and the propagating shock-wave fronts for \mathcal{B}_T^4 , as shown in Fig.1, may offer a distant resemblance to Planck-scale mixmaster dynamics in \mathcal{B}_M^4 .

3 The $\bar{\mathcal{B}}_T^4$ Taub string with effective torsion

A priori, L_o can be equally-well identified with Planck length or with Hubble radius, but in the former case the notion of a regime of scales ‘much larger’ than L_o can be made physically and mathematically relevant. Our elemental ‘quantum-of-time’ $\delta t \sim L_o$, equivalently taken as a $\delta u = L'_o$, can then be formally treated as an ordinary-calculus differential du , allowing the reduction of \mathcal{B}_T^4 to its *effective* $\bar{\mathcal{B}}_T^4$ by a well-defined averaging process: at this limit, differences between adjacent values of $n \in Z$ are negligible, $n \in Z$ is treatable as a continuous $N \in \mathbb{R}$ and summations over n as normalized integrations over u , to give us ⁶

$$\bar{b}^2 := \frac{1}{L'_o} \int_{-L'_o/2}^{L'_o/2} b^2 du = \frac{2}{3B}, \quad \bar{c}^2 := \frac{1}{L'_o} \int_{-L'_o/2}^{L'_o/2} c^2 du = \frac{\pi - 2}{B}, \quad (3.1)$$

$$\overline{(\bar{b}^2)} = 0, \quad \overline{(\bar{c}^2)} = 0; \quad \overline{[(\bar{b}^2)]^2} = \frac{(\pi - 2)B^2}{2L_o^2}, \dots; \quad \overline{(\bar{b}^2)^{..}} = \frac{B}{L_o^2}, \quad \overline{(\bar{c}^2)^{..}} = -\frac{2B}{L_o^2}; \dots \quad (3.2)$$

as results which follow from (2.6) etc. In spite of the presence of $L_o = BL'_o/2$, these results hold only in the higher-scale differences of the $\delta u \gg L'_o$ regime, and, in spite of the $\overline{(\bar{b}^2)} = \overline{(\bar{c}^2)} = 0$ vanishings, remnants of the original dynamics in \mathcal{B}_T^4 effectively do survive in $\bar{\mathcal{B}}_T^4$. To uncover them, we use (3.1) in (1.3) to find the metric of this *static* $\bar{\mathcal{B}}_T^4$ as ⁷

$$d\bar{s}^2 = -2(du)\ell^3 + \bar{b}^2[(\ell^1)^2 + (\ell^2)^2] + \bar{c}^2(\ell^3)^2, \quad \left[\bar{g}_{03} = -1, \quad \bar{g}_{11} = \bar{g}_{22} = \frac{2}{3B}, \quad \bar{g}_{33} = \frac{\pi - 2}{B} \right] \quad (3.3)$$

with $\bar{b}^2 = \bar{b}^2$ and $\bar{c}^2 = \bar{c}^2$. All derivatives of \bar{b}^2, \bar{c}^2 vanish, of course, in sharp contrast to the results in (3.2). We thus find (with $\bar{a}^2 := \bar{c}^2 - \bar{b}^2$)

$$\bar{\Gamma}^\mu{}_\nu = \frac{1}{2L_o\bar{b}^2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\ell^2 & 0 & -\ell^0 - (\bar{b}^2 - \bar{a}^2)\ell^3 & \bar{c}^2\ell^2 \\ \ell^1 & \ell^0 + (\bar{b}^2 - \bar{a}^2)\ell^3 & 0 & -\bar{c}^2\ell^1 \\ 0 & -\bar{b}^2\ell^2 & \bar{b}^2\ell^1 & 0 \end{bmatrix}, \quad \bar{a}^2 := \bar{c}^2 - \bar{b}^2, \quad (3.4)$$

$$\bar{R}^\mu{}_\nu = \frac{1}{(2L_o\bar{b}^2)^2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ \ell^1 \wedge [\ell^0 - \bar{a}^2\ell^3] & 0 & -2\bar{a}^2\bar{b}^2\ell^1\ell^2 & -\bar{c}^2\ell^1 \wedge [\ell^0 - \bar{a}^2\ell^3] \\ \ell^2 \wedge [\ell^0 - \bar{a}^2\ell^3] & -2\bar{a}^2\bar{b}^2\ell^2\ell^1 & 0 & -\bar{c}^2\ell^2 \wedge [\ell^0 - \bar{a}^2\ell^3] \\ 0 & -\bar{b}^2[\ell^0 - \bar{a}^2\ell^3] \wedge \ell^1 & -\bar{b}^2[\ell^0 - \bar{a}^2\ell^3] \wedge \ell^2 & 0 \end{bmatrix},$$

⁶Clearly distinct from the classical limit $L_o \rightarrow 0$, the ‘de-quantization’ limit $L'_o \rightarrow du$ equips the entire process with calculus-approachable dynamics, by which $\bar{\mathcal{B}}_T^4$ can be fully treated as a differentiable manifold.

⁷The bar in $d\bar{s}^2$ and over quantities like \bar{b} , $\bar{\Gamma}^\mu{}_\nu$ or $\bar{\mathcal{R}}^\mu{}_\nu$ is only a marker associating them to $\bar{\mathcal{B}}_T^4$. For averaging we overline, as with $\bar{b}^2, \overline{[(\bar{b}^2)]^2}$, so expressions like those involved in (3.5) below are well defined.

By these results, which include an identically vanishing Weyl tensor, we conclude that, in comparison with the original \mathcal{B}_T^4 , the shock-wave disturbance has disappeared, as expected, but the now conformally flat $\bar{\mathcal{B}}_T^4$ is no-longer Ricci flat. For a closer comparison between the curvatures $R^\mu{}_\nu$, $\bar{R}^\mu{}_\nu$ of \mathcal{B}_T^4 , $\bar{\mathcal{B}}_T^4$ we must resort to the calculation of $\overline{R^\mu{}_\nu}$ directly from (2.3-2.5), knowing beforehand that $R^\mu{}_\nu$ is already Ricci flat, hence it will certainly have to remain so after the averaging. Of course, due to the loss of Ricci flatness in $\bar{\mathcal{B}}_T^4$, we necessarily have $\bar{R}^\mu{}_\nu \neq \overline{R^\mu{}_\nu}$, so the emergence of effective stress-energy content has already occurred in $\bar{\mathcal{B}}_T^4$. If this content is due to effective torsion \bar{T}^μ , that inequality can turn into a precise relation between \mathcal{B}_T^4 and $\bar{\mathcal{B}}_T^4$. Resorting to their basic observables, namely their metrics $g_{\mu\nu}$, $\bar{g}_{\mu\nu}$ and their curvatures $R^\mu{}_\nu$, $\bar{\mathcal{R}}^\mu{}_\nu$ (now allowing for a general connection in $\bar{\mathcal{B}}_T^4$), all expressed in the same time-independent left-invariant frame ℓ^μ , we can then demand that

$$\bar{g}_{\mu\nu} = \overline{g_{\mu\nu}} \ , \quad \bar{\mathcal{R}}^\mu{}_\nu := \bar{R}^\mu{}_\nu + \bar{D}\bar{K}^\mu{}_\nu + \bar{K}^\mu{}_\rho \wedge \bar{K}^\rho{}_\nu = \overline{R^\mu{}_\nu} \ , \quad (3.5)$$

The first requirement is already satisfied in view of (3.3). If $\overline{R^\mu{}_\nu}$ is known or calculable, we can integrate the second to find $\bar{T}^\mu = \bar{K}^\mu{}_\nu \wedge \ell^\nu$. The emergence of this torsion is spontaneous because it is a direct consequence and effective remnant of the original dynamics in \mathcal{B}_T^4 (the full calculation of \bar{T}^μ will be given elsewhere). In any case, with or without effective torsion, the thus-emerging effective stress-energy tensor in $\bar{\mathcal{B}}_T^4$ can be simply found via (3.4) from the Ricci tensor $\bar{R}_{\mu\nu} = \kappa'^2 \bar{T}_{\mu\nu}$, in terms of a Planck-scale κ' length parameter.

4 Finite Taub strings and the \mathcal{B}_M^4 , $\bar{\mathcal{B}}_M^4$ cases

In the second half of this section we will have to lean on speculation, with the mathematical rigor accordingly reduced. Finite Taub strings, open or closed, with any number n of \mathcal{T} elements, can also be realized as stable non-singular and geodesically complete 4D vacua. If they involve a relatively small number of \mathcal{T} elements (like $n = 1, 2, 3, \dots$), they can be viewed as elemental 4D bits taken off the \mathcal{B}_T^4 string vacuum. Their geodesic completeness can be established either by joining the free Σ boundaries of that bit to create a closed string, or extend its geodesics to infinity exactly as with the Taub-NUT completion. Thus, each one of the closed-string $n = 1, 2, 3, \dots$ cases closes upon itself with no links to infinity as $/\mathcal{T}\backslash$, $/\mathcal{T} \vee \mathcal{T}\backslash$, $/\mathcal{T} \vee \mathcal{T} \vee \mathcal{T}\backslash, \dots$, where the initial and final slashes stand for the two free Σ boundaries (as the two halves of a \vee) to be identified. The corresponding open-string cases involve the $\text{NUT} \vee \mathcal{T} \vee \text{NUT}$, $\text{NUT} \vee \mathcal{T} \vee \mathcal{T} \vee \text{NUT}$, $\text{NUT} \vee \mathcal{T} \vee \mathcal{T} \vee \mathcal{T} \vee \text{NUT}$, \dots configurations, along with all the features and charges inherited after the paradigm of the original Taub-NUT completion [9]. For variant types of \mathcal{B}_T^4 vacua, we observe that junctions across the round- S^3 SL sections $du = 0$ at the $b^2 = c^2$ positions of isotropy might also be possible (subject to stability prerequisites), now involving the mentioned Θ_0 normal with $\mathcal{K} = -d\Theta_0$.

The \mathcal{B}_M^4 of the Misner-case, viewable as a 4D string in the present context (but cf. also [9]), could conceivably replace the prototype \mathcal{B}_T^4 (accordingly reduced to a sub-case), but one might also conjecture on a reverse possibility, as we briefly will. In any case, \mathcal{B}_M^4 may well carry fundamental physical content in its Planck-scale mixmaster dynamics. Its principal (a, b, c) radii behave more like variables which can violate Bell's inequalities, just like quantum mechanical ones, with severe restrictions on their causal transforms (like Fourier expansions) and in contrast to the fully Fourier-expandible periodic (b, b, c) radii of \mathcal{B}_T^4 . In likewise sharp contrast to the $\mathcal{B}_T^4, \bar{\mathcal{B}}_T^4$ pair would be any attempt to reconstruct the dynamics of \mathcal{B}_M^4 from excitations over $\bar{\mathcal{B}}_M^4$. The latter has to be a (3+3) type, due to the overall equal participation of its a, b, c radii, with $d\bar{s}^2 = \eta_{\mu\nu}\ell^\mu\ell^\nu$, easily calculable $\bar{R}^\mu{}_\nu$ curvature 2-form and effective stress-energy tensor from $\bar{R}_{\mu\nu} = \kappa'^2\bar{T}_{\mu\nu}$. However, the study of \mathcal{B}_M^4 itself is an entirely different enterprise, which might be facilitated by means of the probabilistic involvement of local Taub breathing-mode dynamics⁸. To further illustrate the point with a related aspect, we recall the mentioned ‘loss of memory’ peculiarity with s-c geodesics, which allows the violation of classical axioms with, e.g., bifurcations in \mathcal{B}_T^4 (to grow much worse in \mathcal{B}_M^4). These, combined with the mentioned presence of geodesics trapped in the potential wells, involve an energy uncertainty $\delta E \sim |V_o| \geq \sqrt{2}/B$ from the s-c depth of those wells. This uncertainty, multiplied by the conjugate one from the $\delta u = \sqrt{2}L_o/B$ width of the wells (our ‘quantum of time’), produces a phase-space surface element of $(\delta E)(\delta u) \geq 2L_o/B^2$. Normalized as $(\delta E)(\delta u) \geq \frac{1}{2}$, this result fixes the time scaling with $B = 2$, hence with $L'_o = L_o$ as in Fig.1, in geometrized $L_o = 1$ units.

5 Discussion

The fundamental scale of any infinitely-long string can be identified as larger or smaller, pending on physical interpretation, as with the L_o scale of \mathcal{B}_T^4 which is equally-well identifiable with Planck length or the Hubble radius H_o , but *that* has little or nothing to do with hierarchy. However, due to the presence and physical content of a second scale in \mathcal{B}_T^4 , the κ_o , we also have a natural hierarchy involved, in the following sense. The scale of κ_o cannot significantly exceed Planck length, regardless of the value of L_o , because, according to (2.7), the string tension would then weaken to the point of making \mathcal{B}_T^4 unstable against large excitations. Thus, κ_o can render L_o largest as a Hubble radius $H_o \gg \kappa_o$, which would place \mathcal{B}_T^4

⁸ As numerical findings indicate [10], the (a, b, c) principal radii in \mathcal{B}_M^4 seem to form patterns of Taub pairings, like a (b, b, c) swaying after a while to a (b, c, b) , and so on. If so, \mathcal{B}_T^4 could help illuminate Planck-scale mixmaster dynamics in the sense that the entire time-evolution of a sufficiently small neighborhood in a homogeneous section of \mathcal{B}_M^4 could be statistically simulated by (b, b, c) , (b, c, b) , or (c, b, b) pairings in an accordingly small neighborhood of the corresponding \mathcal{B}_T^4 and with equally-shared overall participation.

in the rank of a cosmological model, or smallest as $\kappa_o \sim L_o$, which would render \mathcal{B}_T^4 pertinent to Planck-scale dynamics. In both cases, stability and hierarchy stem from the underlying structure of the quantized flow of time, established by the transition of $N \in \mathbb{R} \rightarrow n \in Z$. However, for Planck-length L_o , we also have the availability of ‘de-quantization’ or averaging by going to the reverse limit $n \in Z \rightarrow N \in \mathbb{R}$, as seen. By this we have not, of course, been taken back to the \mathcal{T} we started from, but to the static $\bar{\mathcal{B}}_T^4$ and its effective enrichment. With $\bar{\mathcal{B}}_T^4$ as ground state, excitations thereof must involve two new independent scales, κ and $L_1 \gg L_o$, in addition to κ_o , L_o . We will now briefly expand on some of these aspects.

By the results in section 2, the stability in \mathcal{B}_T^4 relates to the potential-well dynamics from $V(u)$ in-between the Σ_n . The extension of this stability across the Σ_n has emerged from aspects which have sequentially identified the Σ_n as boundaries of a (cured) physical singularity, as junctions and as a gravitational shock-wave front, all relating to the fundamental κ_o -scale string tension on the (0,3) plane along the null ∂_u , L_3 or the non-null E_0 , E_3 vectors. The ‘coincidence’ of identical components (2.7) in so diverse frames is simply due to the restrictive aspects of a traceless $S_{\mu\nu}$ layer on the (0,3) plane. The physical content of this $S_{\mu\nu}$, essentially involved in qualifying \mathcal{B}_T^4 as a 4D string, must be bestowed upon the string tension as a primitive geometric notion. It must be accordingly transferred to the effective torsion \bar{T}^μ , along with the novel type of physical interpretation of the latter in $\bar{\mathcal{B}}_T^4$.

The propagating (∂_u, L_3) planes must all have the correct orientation, because that relates to the direction of propagation of the shock wave through the oriented Σ_n surfaces. In turn, the sign of the normal $\mathcal{N} = L_3 = c\Theta_3$ to the Σ_n relates to the direction of propagation of time and it is carried over, via $d\mathcal{N} \sim \mathcal{K} \sim (c^2)$, to the correct sign of the $S_{\mu\nu}$ stress-energy layers in (2.7) for stability. Reversal of that orientation, even at one junction, would change that sign and apparently destroy the stability of the entire manifold, so the underlying invariance is obviously a fundamental one. The so resulting direction of time is in apparent accord with its other fundamental aspects, namely its quantized flow at Planck scale and its non-holonomic involvement beyond that. The latter emerges in relation to the (e^μ, E_ν) non-singular orthonormal frames in (2.2), and could forbid time reversal for processes involving lengths and times of sufficiently larger-than-Planck scales.

Elements of the mentioned inter-dependences survive in $\bar{\mathcal{B}}_T^4$, and they can be expected to also survive beyond the vacuum geometry. By the holonomy theorems and the Cartan structure equations for any \mathcal{R}_N^M, T^M set, the scale of torsion is completely independent from the scale of the Riemannian part R_N^M of the curvature [7], so excitations off that geometry must be of the Palatini type (variations of metric *plus* connection, independently). As a result, they must necessarily involve, beyond κ_o and L_o , the gravitational coupling κ from excitations of the metric and the independent L_1 from excitations of the connection. This can be applied for the $SU(2) \times U(1)$ K-K manifold, calculated, not over the conventional

$\mathcal{M}_o^4 \times S^3 \times S^1$ ground state (which is not stable and it is not even Ricci flat), but over $\bar{\mathcal{B}}_M^4 \times \mathbb{R}^3 \times S^1$. Thus, the basic difference with the latter is stability and the presence of the effective torsion \bar{T}^μ , which makes it compatible with the effective vacuum equations. The result uncovers the higgsless emergence of the correct EW gauge-boson mass term from variation of the connection, whereby L_1 can be identified with the electroweak scale [11].

Relating to issues on cosmic dynamics and mixing (such as isotropization, homogenization and the horizon problem) in relativistic astrophysics and cosmology, we may re-use (1.1), now with L_o taken as a Hubble radius H_o . \mathcal{B}_T^4 is then identified as an empty cosmological model in breathing mode which re-generates itself eternally. Likewise, the static $\bar{\mathcal{B}}_T^4$ and $\bar{\mathcal{B}}_M^4$ are non-singular empty universes with SL sections, which are squashed or round S^3 . The torsion therein should now be viewed as primordial rather than effective. The effective stress-energy content will be again involved as $\bar{R}_{\mu\nu} = \kappa'^2 \bar{T}_{\mu\nu}$, now with $\kappa' = \kappa$. These models are stable against perturbation and may thus relate to galactic dynamics and to the dark-matter/dark-energy content of the universe.

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